Corollary Let λ be a finite Borel measure on $\mathbb{R}^d$.

(1) Let $A \subset \mathbb{R}^d$ be Borel measurable. Then

$$\text{①} \quad \boxed{\lim_{r \downarrow 0} \frac{\lambda(A \cap B(x,r))}{\lambda(B(x,r))} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}}$$

holds for λ-a.a. $x \in \mathbb{R}^d$.

(2) Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty, \infty\}$ be locally integrable w.r.t. λ. (That is: $\forall x, \exists r, \int_{B(x,r)} f(x) \, d\lambda(x) < \infty$)

Then for λ-a.a. $x \in \mathbb{R}^d$:

$$\text{②} \quad \boxed{\lim_{r \downarrow 0} \frac{1}{\lambda(B(x,r))} \cdot \int_{B(x,r)} f(y) \, d\lambda(y) = f(x)}$$

Proof of the Corollary First we prove that (2) $\Rightarrow$ (1). Namely, let $f = \mathbb{1}_A$.

Now we prove (2). W.l.g. (Without loss of generality) We may assume that $f \geq 0$. Let $\mu(A) := \int_A f(x) \, d\lambda(x)$. Clearly $\mu \ll \lambda$. Moreover, it follows from part (2) of the previous theorem that

$$\text{③} \quad \int_B D(\mu, \lambda, x) \, d\lambda(x) \overset{\downarrow}{=} \mu(B) = \int_B f(x) \, d\lambda(x),$$

for any Borel set $B \subset \mathbb{R}^d$. From the mod 0 uniqueness of the Radon-Nikodym derivative: it follows from ③ that $D(\mu, \lambda, x) = f(x)$ for λ-a.a. $x \in \mathbb{R}^d$. This completes the proof of (2).

Corollary of the second part of the Corollary

Let λ be a finite Borel measure as in the Corollary above and let $f: \mathbb{R}^d \to \mathbb{R}$ be a locally λ-integrable function. Then

$$④ \quad \lim_{r \downarrow 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\lambda(y) = 0.$$

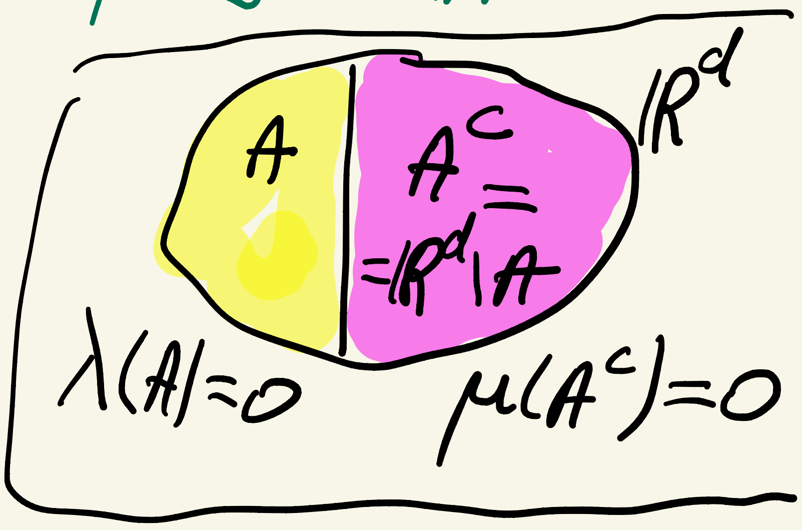
Remark

If $\lambda = \mathcal{L}^d$ then the same proof but using Vitali covering thm for Leb. measure instead of Vitali covering thm for Radon measures yields a stronger statement: Let f be locally $\mathcal{L}^d$-integrable. Then

integration for $\mathcal{L}^d$

$$\lim_{\delta \downarrow 0} \sup \left\{ \left| \frac{\int_B f(x) dx}{\mathcal{L}^d(B)} - f(x) \right| : \begin{array}{l} B \text{ is a ball with } x \in B \\ \text{and } |B| < \delta \end{array} \right\} = 0.$$

Recall: We say that the Radon measures λ & μ are singular ($\lambda \perp \mu$) if $\exists\, A \subset \mathbb{R}^d$ s.t. $\lambda(A) = 0$ & $\mu(\mathbb{R}^d \setminus A) = 0$.



The following thm is a composition Radon-Nikdym thm & Leb. decomposition theorem.

Theorem Let λ, μ be finite Radon measures on $\mathbb{R}^d$. Then there exists a Borel function f and a Radon measure ν s.t.

(1) $\lambda \perp \nu$,

(2) $\mu(B) = \int_B f \, d\lambda + \nu(B)$,

(3) $\mu \ll \lambda \iff \nu = 0$.

Proof Let $A := \{x \in \mathbb{R}^d : \underline{D}(\mu, \lambda, x) < \infty\}$. Let

$$\mu_1 := \mu|_A \quad \& \quad \nu := \mu|_{\mathbb{R}^d \setminus A}.$$

Recall that we have proven above the following: let μ be a finite Borel measure on $\mathbb{R}^d$ & let $H \subset \mathbb{R}^d$ then

$$\lim_{r \downarrow} \frac{\mu(B(x,r) \cap H)}{\mu(B(x,r))} = 1 \text{ for } \mu\text{-a.e. } x.$$

We prove that $\mu_1 \ll \lambda$. We know that $\forall x \in A$, $\underline{D}(\mu_1, \lambda, x) < \infty$.

This implies that $\mu_1 \ll \lambda$. Namely,

$$\frac{\mu(B(x,r) \cap A)}{\lambda(B(x,r))} \le \frac{\mu(B(x,r))}{\lambda(B(x,r))} \quad \forall x \in A$$

$\underline{D}(\mu_1, \lambda, x) < \infty$ $\qquad$ $\underline{D}(\mu, \lambda, x) < \infty$

On the other hand, for all $x \in A^c = \mathbb{R}^d \setminus A$ we have by definition $\underline{D}(\mu, \lambda, x) = \infty$.

We know as mentioned above: μ-a.a. $x \in \mathbb{R}^d \setminus A$

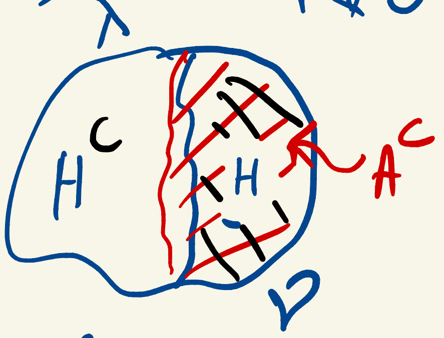
$\exists H \subset A^c$, $\mu(A^c \setminus H) = 0$

$$\lim_{r \downarrow 0} \frac{\mu(B(x,r) \cap A^c)}{\mu(B(x,r))} = 1. \text{ s.t. } \forall x \in H \text{ we have this}$$

For an $x \in H$:

$$\lim_{r \downarrow 0} \frac{\nu(B(x,r))}{\lambda(B(x,r))} = \lim_{r \downarrow 0} \frac{\mu(B(x,r) \cap A^c)}{\lambda(B(x,r))} = \lim_{r \downarrow 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} = \infty.$$

That is $\forall x \in H$ $\underline{D}(\nu, \lambda, x) = \infty$



$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap A^c)}{\lambda(B(x,r))}$$

$x \in H$ $\longrightarrow 1$

If $\nu \neq 0$ then

$\underline{D}(\nu, \lambda, x) = \infty$ for μ-a.a. $x \in A^c$; $\qquad H \subset A^c$

$\mu(A^c \setminus H) = 0$

$H \subset \{x \in A^c : \underline{D}(\nu, \lambda, x) = \infty\}$

$\forall x \in H$ $\quad \overline{D}(\nu, \lambda, x) \ge t \Rightarrow \nu(H) \ge t \cdot \lambda(H)$ $\quad \forall t$

$\boxed{\lambda(H) = 0} \Rightarrow \boxed{\nu \perp \lambda}$.

Density Theorems for Hausdorff & Packing measures

Definition Let $0 \le s < \infty$, $A \subset \mathbb{R}^d$, $a \in A$. The upper & lower s-densities of A at a are

$$\overline{d}^s(A,a) := \overline{\lim_{r \downarrow 0}} \frac{1}{(2r)^s} \mathcal{H}^s(A \cap B(x,r)),$$

$$\underline{d}^s(A,a) := \underline{\lim_{r \downarrow 0}} \frac{1}{(2r)^s} \mathcal{H}^s(A \cap B(x,r)).$$

If $\overline{d}^s(A,a) = \underline{d}_s(A,a)$ then this common value is called the s-dimensional density of A at a & we denote it by $d^s(A,a)$.

The following theorem resembles to the Leb. density theorem.

Theorem Let $A \subset \mathbb{R}^d$ with $\mathcal{H}^s(A) < \infty$. Then

(1) $2^{-s} \le \overline{d}^s(A,a) \le 1$ for $\mathcal{H}^s$-a.a. $x \in \mathbb{R}^d$

(2) If $A \subset \mathbb{R}^d$ is $\mathcal{H}^s$-measurable then $\overline{d}^s(A,a) = 0$ for $\mathcal{H}^s$-a.a. $x \in \mathbb{R}^d \setminus A$.

Corollary Let $A, B \subset \mathbb{R}^d$ be $\mathcal{H}^s$-measurable with $B \subset A$ & $\mathcal{H}^s(A) < \infty$. Then for $\mathcal{H}^s$-a.a. $x \in B$
$\overline{d}^s(B,x) = \overline{d}^s(A,x)$ & $\underline{d}^s(B,x) = \underline{d}^s(A,x)$.

Proof of the Corollary We apply part (2) for the set $A \setminus B$.

Remark There exists $C \subset \mathbb{R}^d$ compact with $0 < \mathcal{H}^s(C) < \infty$, $\underline{d}^s(C,x) = 0$, $\forall x \in \mathbb{R}^d$.

Proof of the Theorem

The proof of the first inequality of (1):

$B = \{x \in A : \overline{d}^s(A, x) < 2^{-s}\}$. Then $B = \bigcup_k B_k$, where

$B_k = \{x \in A : \mathcal{H}^s(A \cap B(x, r)) < \frac{k}{k+1} \cdot r^s$ for $0 < r < \frac{1}{k},\ k = 1, 2, \dots\}$

It is enough to prove that $\mathcal{H}^s(B_k) = 0,\ \forall k$.

To see this fix a k and let $t = \frac{k}{k+1}$ and $\varepsilon > 0$.

We can find a $\frac{1}{k}$-cover $\{E_i\}$ of B_k s.t.

$B_k \subset \bigcup_i E_i,\ |E_i| < \frac{1}{k},\ B_k \cap E_i \neq \emptyset,\ \sum_i |E_i|^s < \mathcal{H}^s(B_k) + \varepsilon$. (*)

For every i, let $x_i \in B_k \cap E_i$ and $r_i = |E_i|$. Then

$B_k \cap E_i \subset A \cap B(x_i, r_i)$ and

$x_i \in B_k \Rightarrow \mathcal{H}^s(A \cap B(x_i, r)) < t \cdot r^s$, $r < \frac{1}{k}$

$$\mathcal{H}^s(B_k) \leq \sum_i \mathcal{H}^s(B_k \cap E_i) \leq \sum_i \mathcal{H}^s(A \cap B(x_i, r_i)) < \sum_i t \cdot r_i^s$$

$$\leq t \cdot \sum_i |E_i|^s < t(\mathcal{H}^s(B_k) + \varepsilon).$$

by (*) above

Since $B_k \subset A$ & by assumption $\mathcal{H}^s(A) < \infty$

Now we let $\varepsilon \downarrow 0$ and get $\mathcal{H}^s(B_k) \leq t \cdot \mathcal{H}^s(B_k)$

Note that by the facts: $t < 1$ & $\mathcal{H}^s(B_k) < \infty$.

this is possible only if $\mathcal{H}^s(B_k) = 0$. (**)

This implies that $\mathcal{H}^s(B)=0$. That is the proof of the left hand-side inequality in (1) is completed.

Now we verify the right hand-side inequality in (1). Using that $\mathcal{H}^s$ is Borel regular (Theorem A from Covering Theorems II.pdf.) now we may assume that A is a Borel set.

Let $t>1$, $B:=\{x\in A: \overline{d}^s(A,x)>t\}$. It is enough to show that (***) $\mathcal{H}^s(B)=0$.

Let $\varepsilon,\delta>0$. We know that $\mathcal{H}^s|_A$ is a Radon measure (Theorem A from Covering Theorems II.pdf). Hence we can choose an open set U s.t

$$B\subset U \ \& \ \mathcal{H}^s(A\cap U)<\mathcal{H}^s(B)+\varepsilon$$

For $\forall x\in B$ $\exists$ arbitrarily small $0<r<\frac{\delta}{2}$ s.t.

$B(x,r)\subset U$ & $\mathcal{H}^s(A\cap B(x,r))>t\cdot(2r)^s$ (by the def of B).

Let $\mathcal{B}$ be the collection of these balls. Then $\mathcal{B}$ satisfies the assumptions of the Vitaly Covering Thm for Radon measures ($\mathcal{H}^s|_A$ is a Radon measure).

Hence $\exists$ a disjoint $\{B_i\}_{i=1}^{\infty}\subset\mathcal{B}$ s.t.

$$\mathcal{H}^s(B\setminus\bigcup_i B_i)=0.$$

$$\mathcal{H}^s(B)+\varepsilon > \mathcal{H}^s(A\cap U) \underset{\substack{\{B_i\}\text{ disj.}\\ B_i\subset U}}{\geq} \sum_i \mathcal{H}^s(A\cap B_i) \underset{\text{def of } B}{>} t\sum_i |B_i|^s$$

$$\underset{r\leq \frac{\delta}{2}}{\geq} t\,\mathcal{H}^s_\delta\Big(B\cap \bigcup B_i\Big) \doteq t\cdot \mathcal{H}^s_\delta(B)$$

This step is explained below

Namely,

Since $0=\mathcal{H}^s(B\setminus \bigcup_i B_i)=\sup_{\delta>0}\underbrace{\mathcal{H}^s_\delta(B\setminus \bigcup_i B_i)}_{=0}$

$$\mathcal{H}^s_\delta(B)\leq \overbrace{\mathcal{H}^s_\delta\Big(B\setminus \bigcup_i B_i\Big)}^{0}+\underbrace{\mathcal{H}^s_\delta\Big(B\cap \bigcup_i B_i\Big)}_{\leq \mathcal{H}^s_\delta(B)}$$

in fact $\mathcal{H}^s_\delta(\cdot)$ is an outer measure but NOT a metric outer measure

$\mathcal{H}^s_\delta(\cdot)$ is sub-additive

$$\mathcal{H}^s\Big(B\setminus\bigcup_i B_i\Big)=\sup_{\delta>0}\mathcal{H}^s_\delta\Big(B\setminus \bigcup_i B_i\Big)$$

this is $=0$ since the left hand-side $=0$.

From here $\boxed{\mathcal{H}^s_\delta(B)=\mathcal{H}^s_\delta\Big(B\cap \bigcup_i B_i\Big)}$

So, we got $\mathcal{H}^s(B)+\varepsilon > t\cdot \mathcal{H}^s_\delta(B)$.

Here we let $\varepsilon\downarrow 0$ and then $\delta\to 0$ to get that $\mathcal{H}^s(B)\geq t\,\mathcal{H}^s(B)$. Using that $t>1$ we get that $\mathcal{H}^s(B)=0$. Since $t>1$ was arbitrary this is the proof of the right hand-side inequality of (1).

Now we prove part (2) of the theorem

$A^c := \mathbb{R}^d \setminus A$

Let $t>0$. $B := \{x \in A^c : \overline{D}^s(A,x) > t\}$. We need to prove that $\mathcal{H}^s(B)=0$ (*). Let $\varepsilon>0$. We know that $\mathcal{H}^s|_A(B)=0$. $\exists\, U$ open set s.t.

since $B\subset A^c$. So, $\mathcal{H}^s|_A(A^c)=0$.

$B\subset U$ & $\mathcal{H}^s(A\cap U)<\varepsilon$ (**).

$\mathcal{H}^s|_A$ is a Radon measure

Fix a $\delta>0$.

$\forall x\in B$, $\exists\; 0<r(x)<\frac{\delta}{10}$ s.t.

by the def of B

$B(x,r(x))\subset U$, $\mathcal{H}^s(A\cap B(x,r(x)))>t\,(2\cdot r(x))^s$ (***)

Now we use the $5r$-covering theorem. It yields that $\exists\; x_1,x_2,\ldots\in B$ s.t.

- $B_i := B(x_i, r(x_i))$ are pairwise disjoint
- $\{5B_i\}$ cover B. That is $B\subset \bigcup_i 5B_i$. Then

$$t\mathcal{H}^s_\delta(B)\le t\cdot\sum_i |5B_i|^s = t\,5^s\sum_i |B_i|^s \overset{(***)}{<} 5^s\sum_i \mathcal{H}^s(A\cap B_i)$$

$$\le 5^s\,\mathcal{H}^s(A\cap U) \overset{(**)}{<} 5^s\cdot\varepsilon$$

$\{B_i\}$ disj. $B_i\subset U$

Hence $t\,\mathcal{H}^s_\delta(B)<5^s\varepsilon$ for all $\delta>0$, $\varepsilon>0$

That is $t\cdot\mathcal{H}^s(B)<5^s\varepsilon$ for all $\varepsilon>0$.

Hence $\mathcal{H}^s(B)=0$.

Regular & irregular points & sets

$E \subset \mathbb{R}^d$ is termed an *s-set* $(0 \le s \le d)$ if E is $\mathcal{H}^s$-measurable & $0 < \mathcal{H}^s(E) < \infty$.

Let $x \in E$. We say x is *a regular point of E* if $\underline{D}^s(E,x) = \overline{D}^s(E,x) = 1$. We say that an s-set $E \subset \mathbb{R}^d$ is *regular* if $\mathcal{H}^s$-a.a. $x \in E$ is *regular*. We say that an s-set $E \subset \mathbb{R}^d$ is *irregular* if $\mathcal{H}^s$-a.a. $x \in E$ is *irregular*.

Corollary of the Corollary above

Let $E = \bigcup_j E_j$, E_j is an s-set for all j & E is an s-set. Then for every j

$$\underline{d}^s(E_j,x) = \underline{d}^s(E,x) \quad \& \quad \overline{d}^s(E_j,x) = \overline{d}^s(E,x); \quad \mathcal{H}^s\text{-a.a. } x \in E_j.$$

This implies

Corollary of the Corollary of the Corollary

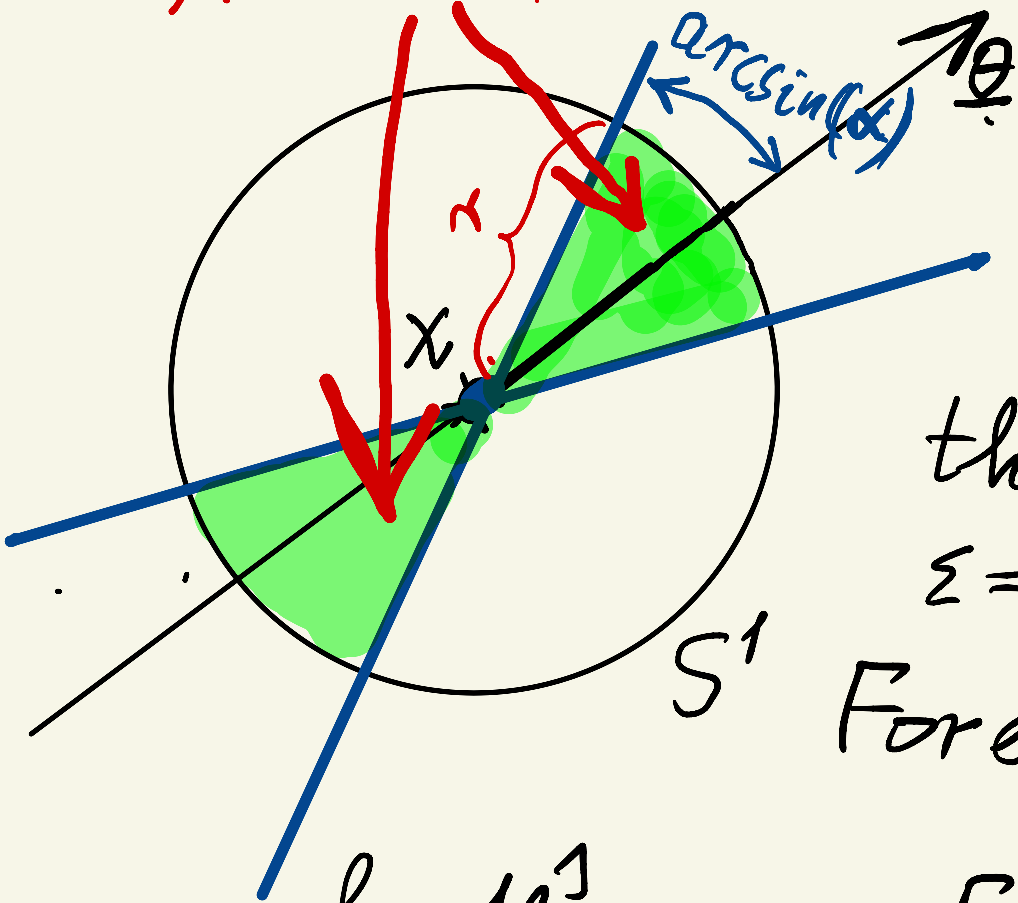
Let $E \subset \mathbb{R}^d$ be an s-set. If E is regular respectively, irregular then any measurable subset of positive measure is regular respectively irregular.

Theorem An s-set $E \subset \mathbb{R}^d$ is irregular unless $s \in \mathbb{N}$.

A Theorem of Mattila

Let $\underline{x}\in\mathbb{R}^2$, $\underline{\theta}\in S^1$ (a unit vector on $\mathbb{R}^2$). $0<\alpha\leq 1$, $r>0$

$$X(x,r,\underline{\theta},\alpha)=\{y\in\mathbb{R}^2 : |\sin\measuredangle(\underline{\theta},y-x)|<\alpha\}\cap B(x,r).$$



Theorem (Mattila 1988)

Let $1<s\leq 2$, $0<\alpha\leq 1$ then $\exists$ a constant $\varepsilon=\varepsilon(s,\alpha)>0$ such that For every $E\subset\mathbb{R}^2$ with $\mathcal{H}^s(E)<\infty$ for $\mathcal{H}^s$-a.e. $x\in E$ we have

$$\limsup_{r\downarrow 0}\ \inf_{\underline{\theta}\in S^1}\frac{\mathcal{H}^s(E\cap X(x,r,\underline{\theta},\alpha))}{(2r)^s}\geq\varepsilon.$$